Fine-scale statistics in number theory, geometry and dynamics

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https://people.maths.bris.ac.uk/~majm/bib/talks/ICERM_Marklof2020.pdf; file version June 28, 2020

Here is the plan for our three sessions:

- How can we measure randomness in deterministic sequences?
- From deterministic sequences to random point processes
- Case study 1: Hitting and return times for linear flows on flat tori
- Case study 2: Fractional parts of $\sqrt{n}$
- [Case study 3: Directions in hyperbolic lattices]

How can we measure randomness in deterministic sequences

Gap statistics

- Consider ordered sequence of real numbers

$$0 \le a_1 \le a_2 \le \cdots \to \infty$$

of **density one**, i.e.,

$$\lim_{T\to\infty} \frac{N[0,T]}{T} = 1, \qquad N[0,T] := \#\{n \mid a_n \le T\}.$$

This ensures the average gap between elements in this sequence is 1.

- **Gap distribution**

$$P_T[a,b] = \frac{\#\{n \le N[0,T] \mid a_{n+1} - a_n \in [a,b]\}}{N[0,T]}$$

- The counting measure P_T defines a probability measure on $\mathbb{R}_{\ge 0}$. **Does** P_T **converge (weakly) to some probability measure** P **as** $T \to \infty$**?** I.e.,

$$\lim_{T\to\infty} P_T[a,b] = P[a,b] \quad \forall\, 0 \le a < b < \infty$$

Example: Integers

For

$$a_n = n$$

we have

$$P_T[a,b] = \frac{\#\{n \le N[0,T] \mid 1 \in [a,b]\}}{N[0,T]} = \delta_1[a,b]$$

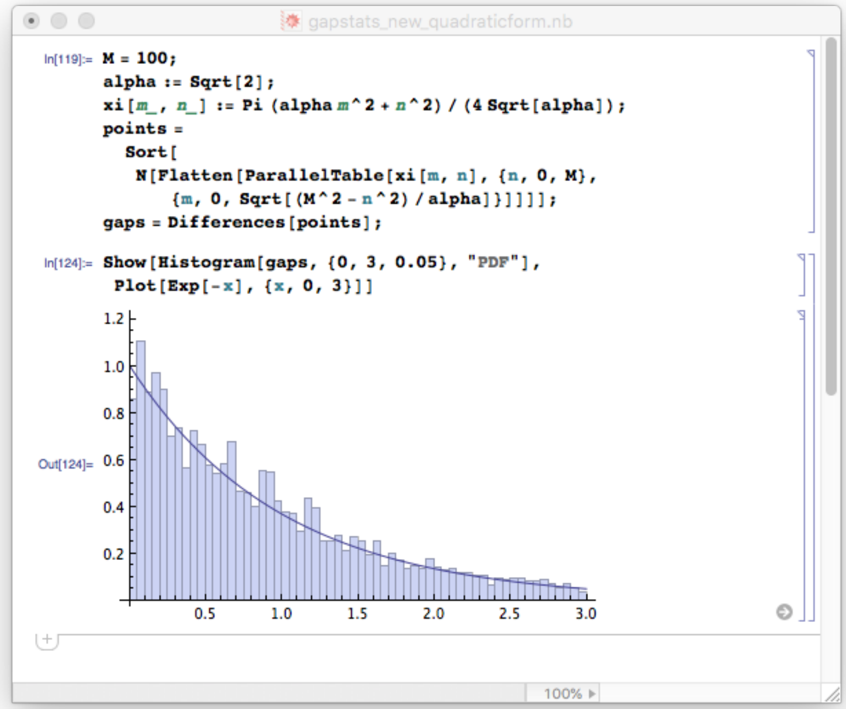
So $P_T = \delta_1 = P$ (the Dirac mass at 1).

Example: Quadratic forms at integer lattice points I*

- Let $(a_n)_n$ given by the set

$$\left\{ \frac{\pi(\alpha m^2 + n^2)}{4\sqrt{\alpha}} \;\middle|\; m, n \in \mathbb{Z}^2_{\geq 0} \right\}$$

- Note $(a_n)_n$ has density one (check!)
- We have no proof $P_T \xrightarrow{\text{w}} P$ with P the exponential distribution for any α
- For $\alpha \in \mathbb{Q}$ one can show $P_T \xrightarrow{\text{w}} \delta_0$



Note: The exponential distribution is the gap distribution ("waiting times") of a Poisson point process of intensity one

*These examples are already discussed M. Berry and M. Taylor, Proc. Roy. Soc 1977 who where interested in energy level statistics in the context of quantum chaos

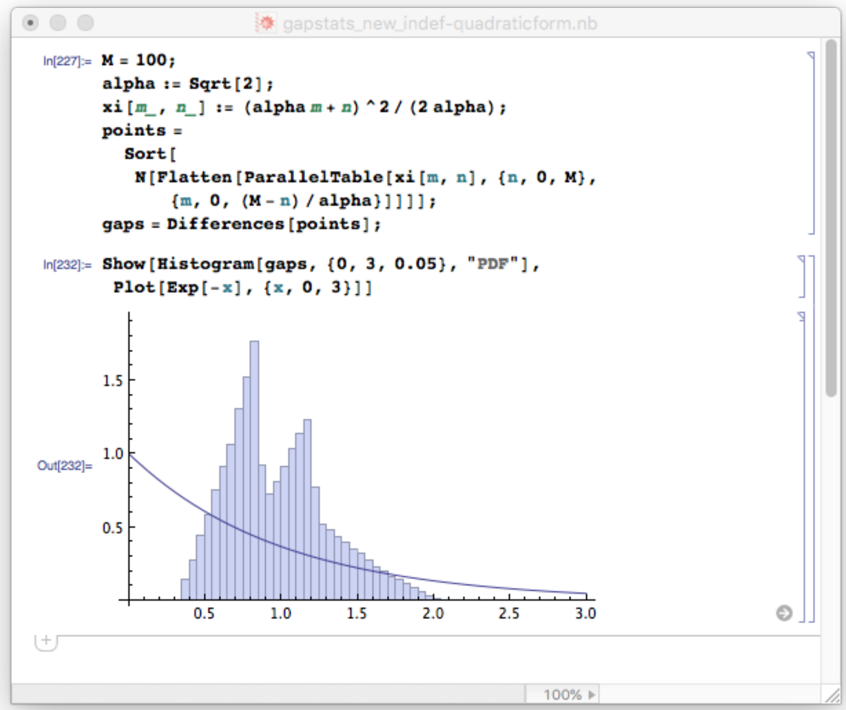
Example: Quadratic forms at integer lattice points II

The previous example was a positive definite quadratic form. How about the following discriminant-zero case:

- Let $(a_n)_n$ given by the set

$$\left\{ \frac{(\alpha m + n)^2}{2\alpha} \,\middle|\, (m,n) \in \mathbb{Z}^2_{\geq 0} \right\}$$

- Note $(a_n)_n$ has density one (check!)
- One can show that P_T does not converge for $\alpha \notin \mathbb{Q}$ (only along subsequences), and understand the distribution in terms of the “three gap theorem”



Exercise 1: Show that for $\alpha \in \mathbb{Q}$ we have $P_T \xrightarrow{\text{w}} \delta_0$.

Example: Quadratic forms at integer lattice points II

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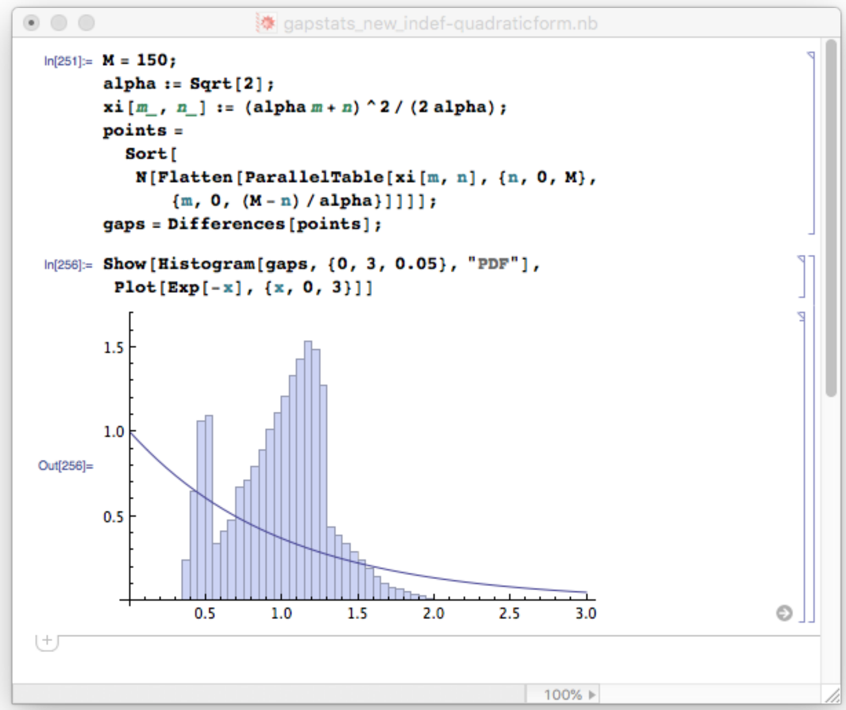

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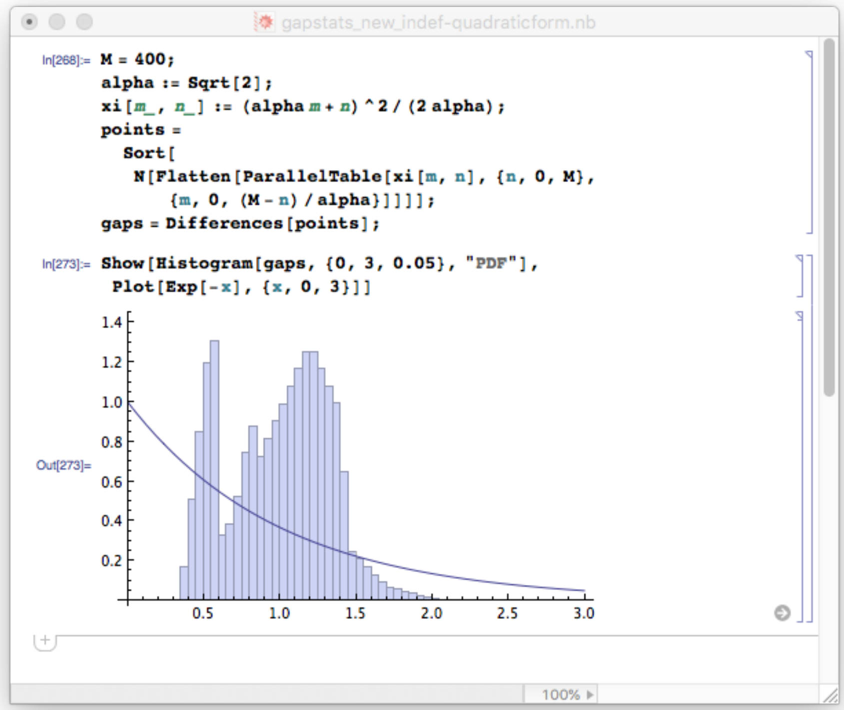


- Note $(a_n)_n$ has density one (check!)
- One can show that P_T does not converge for $\alpha \notin \mathbb{Q}$ (only along subsequences), and understand the distribution in terms of the "three gap theorem"

Exercise 1: Show that for $\alpha \in \mathbb{Q}$ we have $P_T \xrightarrow{\text{w}} \delta_0$.

Rescaling

Suppose the sequence $0 \le a_1 \le a_2 \le \cdots \to \infty$ does not have density one, but satisfies the more general

$$\lim_{T\to\infty} \frac{N[0,T]}{L(T)} = 1, \qquad N[0,T] := \#\{n \mid a_n \le T\}.$$

with the **integrated density** $L(T) = \nu[0,T] = \int_0^T \nu(dt)$ and the Borel measure ν is absolutely continuous with respect to Lebesgue measure dt.

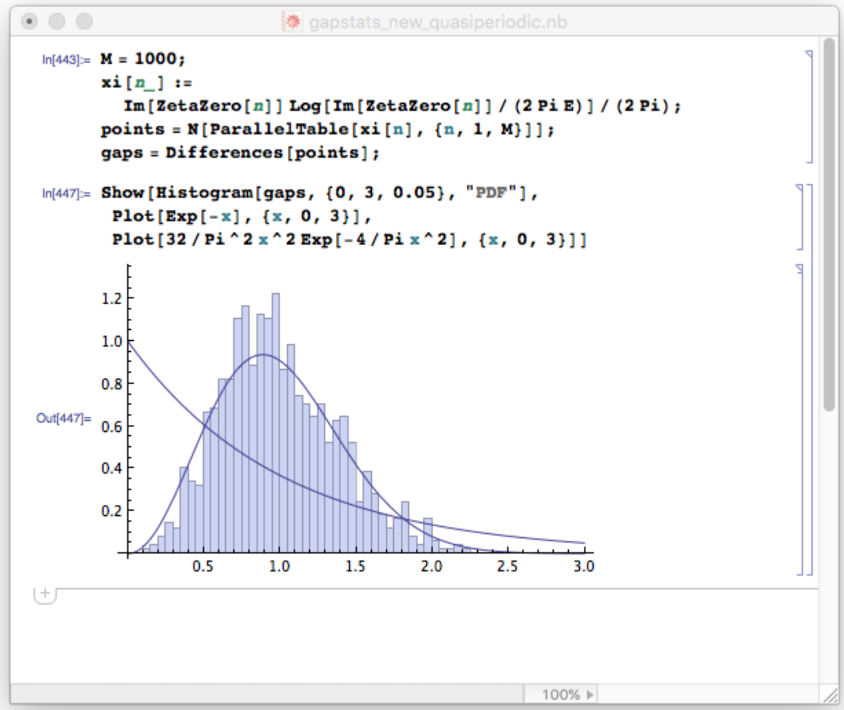
Then the **rescaled sequence** $b_n = L(a_n)$ has density one and it is more natural consider the gap distribution for this rescaled sequence than the "raw" gap distribution the original sequence.

Note $N[0,T] = \sum_n \delta_{a_n}[0,T] = \int_0^T \sum_n \delta_{a_n}(dt)$ so we cannot take $L(T) = N[0,T]$.

Question: What would the gap distribution be for the choice $L(T) = N[0,T]$?

Example: The Riemann zeros

- Let a_n be the imaginary part of the nth Riemann zero on the critical line (in the upper half plane).
- Then a_n has a density given by $L(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e}$.
- Consider gap distribution of the rescaled zeros $b_n = \frac{a_n}{2\pi} \log \frac{a_n}{2\pi e}$.
- We have no proof $P_T \to P$ with P given by the limiting cap distribution for large unitary random matrices.



General fine-scale statistics

- Consider

$$0 \leq a_1 \leq a_2 \leq \cdots \to \infty$$

  of **density one** (as before).
- Fix σ a locally finite Borel measure on $\mathbb{R}_{\geq 0}$ so that $\sigma[0, \infty) = \infty$.
- For $D \subset \mathbb{R}$ a compact interval, set $N(D) = \#\{n \mid a_n \in D\}$ and denote by $t + D$ its translation by t.
- For $k \in \mathbb{Z}_{\geq 0}$

$$E_\sigma([0, T], D, k) = \frac{\sigma\{t \in [0, T] \mid N[t + D] = k\}}{\sigma[0, T]}$$

  is the probability that, for t random in $[0, T]$ (w.r.t. σ), the interval $t + D$ contains k contains elements of $(a_n)_n$

Example: Gap and nearest distance statistics

- Take $D = [0, s]$, $k = 0$, $\sigma = \sum_{n=1}^{\infty} \delta_{a_n}$ (so $A = 1$ by assumption); then

$$E_\sigma([0,T],[0,s],0) = \frac{\#\{n \le N[0,T] \mid N[a_n, a_n + s] = 0\}}{N[0,T]} = P_T[0,s]$$

We have recovered the **gap distribution** of $(a_n)_n$!

- If instead we take $D = [-s, s]$, then

$$E_\sigma([0,T],[-s,s],0) = \frac{\#\{n \le N[0,T] \mid N[a_n - s, a_n + s] = 0\}}{N[0,T]}$$

which is the **nearest distance distribution** of $(a_n)_n$.

Example: Gap and void statistics

- For $\sigma = \mathsf{Leb}$ (the Lebesgue measure normalised so that $\mathsf{Leb}[0,1] = 1$) then

$$E_{\mathsf{Leb}}([0,T], D, 0) = \frac{\mathsf{Leb}\{t \in [0,T] \mid N[t+D] = 0\}}{T}$$

is called the **void distribution** of $(a_n)_n$.

Exercise 2: Show that

for $T > 0$ and $s \in \mathbb{R}_{>0} \setminus \{\text{discont.}\}$ we have

$$\frac{N(T)}{T} P_T[s, \infty) = -\frac{d}{ds} E_{\mathsf{Leb}}([0,T],[0,s],0) \\ - \frac{1}{T}\mathbb{1}(a_1 > s) - \frac{1}{T}\mathbb{1}(a_{N(T)} > T - s)$$

$$\{\text{discont.}\} = \{a_1\} \cup \{a_{n+1} - a_n \mid n \le N(T) - 1\} \cup \{T - a_{N(T)}\}$$

Hint: Work out $E_{\mathsf{Leb}}([a_n, a_{n+1}), [0,s], 0)$.

Example: Pigeon hole statistics

- Take $D = [0, s)$ (e.g. $s = 1$), $k \in \mathbb{Z}_{\geq 0}$, $\sigma = \sum_{n=0}^{\infty} \delta_{ns}$ (so $A = 1/s$); then

$$E_\sigma([0,T],[0,s],k) = \frac{\#\{0 \leq n \leq T/s \mid N[ns,(n+1)s) = k\}}{\lfloor T/s \rfloor + 1}$$

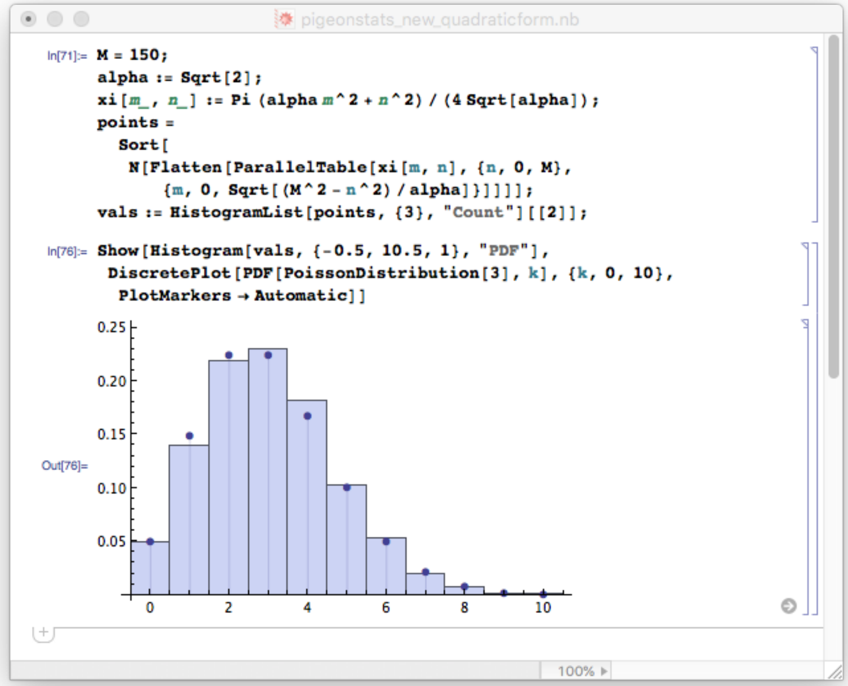
i.e. the proportion of bins $[0,s], [s,2s], [2s,3s], \ldots, [s\lfloor T/s \rfloor, s(\lfloor T/s \rfloor + 1)]$ that contain exactly k points

Example: Quadratic forms at integer lattice points

- Let $(a_n)_n$ given by the set

$$\left\{ \frac{\pi(\alpha m^2 + n^2)}{4\sqrt{\alpha}} \;\middle|\; m, n \in \mathbb{Z}^2_{\geq 0} \right\}$$

- In the experiment we have taken the pigeon hole stats $E_\sigma([0,T],[0,s],k)$ with bin width $s = 3$.



- We expect $\lim\limits_{T\to\infty} E_\sigma([0,T],[0,s],k) = \frac{s^k}{k!}\mathrm{e}^{-s}$ (the Poisson distribution), but no proof

Two-point correlations

- The above local statistics are often too difficult to handle analytically; two-point statistics are more tractable

- **Pair correlation measure**

$$R_T[a,b] = \frac{\#\{(m,n) \mid n \le N[0,T],\ m \ne n,\ a_m - a_n \in [a,b]\}}{N[0,T]}$$

- Compare with gap distribution

$$P_T[a,b] = \frac{\#\{n \le N[0,T] \mid a_{n+1} - a_n \in [a,b]\}}{N[0,T]}$$

- For positive definite quadratic forms, under explicit Diophantine conditions on the coefficients, one can prove* $R_T[a,b] \to b - a$

*A. Eskin, G. Margulis, S. Mozes, Annals Math 2005

Example: Riemann zeros

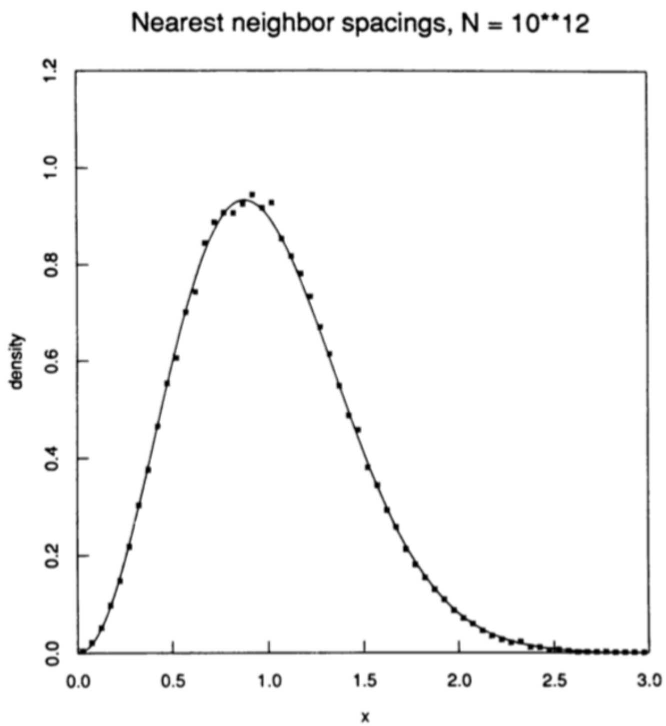


FIGURE 4

Probability density of the normalized spacings δ_n. Solid line: GUE prediction. Scatter plot: empirical data based on zeros γ_n, $10^{12} + 1 \leqslant n \leqslant 10^{12} + 10^5$.

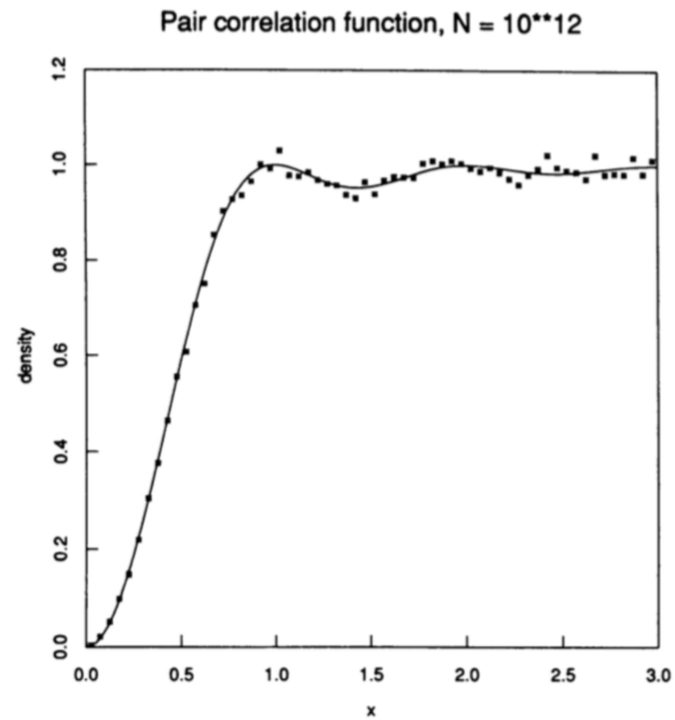


FIGURE 2

Pair correlation of zeros of the zeta function. Solid line: GUE prediction. Scatter plot: empirical data based on zeros γ_n, $10^{12} + 1 \leqslant n \leqslant 10^{12} + 10^5$.

A. M. Odlyzko, Math. Comp. 1987

best result to-date (and a beautiful paper) Z. Rudnick, P. Sarnak, Duke. Math. J. 1996

From deterministic sequences to random point processes

Randomization

- Consider

$$0 \leq a_1 \leq a_2 \leq \cdots \to \infty$$

of **density one** (as before).

- Fix σ a locally finite Borel measure on $\mathbb{R}_{\geq 0}$ so that $\sigma[0,\infty) = \infty$.
- Let t be a random variable distributed on $[0,T]$ with respect to σ; that is t is defined by $\mathbb{P}(t \in B) = \dfrac{\sigma(B \cap [0,T])}{[0,T]}$ for any Borel set $B \subset \mathbb{R}$.
- Define the random point process (=a random counting measure on $\mathbb{R}$)

$$\xi_T = \sum_{n=1}^{\infty} \delta_{a_n - t}$$

- Note: $E_\sigma([0,T], D, k) = \mathbb{P}(\xi_T D = k)$.
- Is there a limiting point process $\xi_T \xrightarrow{\text{d}} \xi$ as $T \to \infty$?

Point processes[*]

- $\mathcal{M}(\mathbb{R})$ the space of locally finite Borel measures on $\mathbb{R}$, equipped with the vague topology[†]
- $\mathcal{N}(\mathbb{R}) \subset \mathcal{M}(\mathbb{R})$ the closed subset of integer-valued measures, i.e., the set of ζ such that $\zeta B \in \mathbb{Z} \cup \{\infty\}$ for any Borel set B
- A **point process** on $\mathbb{R}$ is a random measure in $\mathcal{N}(\mathbb{R})$
- For $\zeta \in \mathcal{N}(\mathbb{R})$, we can write $\zeta = \sum_j \delta_{\tau_j(\zeta)}$ where $\tau_j : \mathcal{N}(\mathbb{R}) \to \mathbb{R}$.
- Use convention $\tau_j \leq \tau_{j+1}$, and $\tau_0 \leq 0 < \tau_1$ if there are $\tau_j \leq 0$.
- ζ is **simple** if $\sup_t \zeta\{t\} \leq 1$ a.s
- The **intensity measure** of ζ is defined as $\mathbb{E}\zeta$.

[*]For general background see O. Kallenberg, Foundations of Modern Probability, Springer 2002

[†]The vague topology is the smallest topology such that the function $\widehat{f} : \mathcal{M}(\mathbb{R}^d) \to \mathbb{R},\ \mu \mapsto \mu f$ is continuous for every $f \in \mathrm{C}_c(\mathbb{R}^d)$.

Example: Poisson point processes

- Fix $\sigma \in \mathcal{M}(\mathbb{R})$
- The **Poisson point process** with intensity measure σ is defined by

$$\mathbb{P}(\zeta B_i = k_i,\ i = 1, \ldots, r) = \prod_{i=1}^{r} \frac{(\sigma B_i)^{k_i}}{k_i!} \mathrm{e}^{-\sigma B_i}$$

  for all bounded and pairwise disjoint Borel sets B_i, integers $k_i \geq 0$, $r > 0$.
- ζ is called **homogeneous** Poisson process if σ is Lebesgue measure.

Exercise 3: Show that σ is indeed the intensity measure of the Poisson point process ζ, i.e. verify $\mathbb{E}\zeta = \sigma$.

Stationarity

- For $u \in \mathbb{R}$, define the **shift operator** θ^u on $\mathcal{M}(\mathbb{R})$ by $\theta^u \zeta B = \zeta(B + u)$ for every Borel set $B \subset \mathbb{R}$.
- If $\zeta = \sum_j \delta_{\tau_j(\zeta)}$, we have $\theta^u \zeta = \sum_j \delta_{\tau_j(\zeta) - u}$,
- A random $\zeta \in \mathcal{M}(\mathbb{R})$ is **stationary** if $\theta^u \zeta \stackrel{\mathrm{d}}{=} \zeta$ for all $u \in \mathbb{R}$.
- The intensity measure of a stationary random measure ζ is $\mathbb{E}\zeta = I_\zeta \mathrm{Leb}$, where the **intensity** is given by $I_\zeta = \dfrac{\mathbb{E}\zeta(0, R]}{R}$, which, by stationarity, is independent of the choice of $R > 0$.

Exercise 4: Show that a homogeneous Poisson point process is stationary.

Example: Hitting times for flows*

- Consider the topological flow $\varphi^t : X \to X$ where (X, ν) is a probability space and ν invariant under φ^t.
- Choose a measurable section $Y \subset X$ that is transversal to the flow, i.e., there is $\epsilon > 0$ such that $\varphi^t Y \cap Y = \emptyset$ for $-\epsilon < t < \epsilon$.
- For $x \in X$, let $(t_j(x))_{j\in\mathbb{Z}}$ be the sequence of **hitting times** (forward and backward in time) given by the ordered point set $\{t \in \mathbb{R} \mid \varphi^t(x) \in Y\}$.
- For x random, $\xi = \sum_j \delta_{t_j(x)}$ defines a **simple random point process**.

Exercise 5: Show that, if x is distributed according to the invariant measure ν, then ξ is a stationary point process.

*Main reference for this section: J. Marklof, Nonlinearity 2019

Example: Hitting times for flows

- Let $S = \bigcup_{t \in \mathbb{R}} \varphi^t(\partial Y)$ be the set of all x that will hit the boundary of Y at least once.

Theorem 1: The map

$$\iota : X \to \mathcal{M}(\mathbb{R}), \qquad x \mapsto \sum_j \delta_{t_j(x)}$$

is continuous on $X \setminus S$.

Proof:

- We need to show that, for every $f \in \mathsf{C}_c(\mathbb{R})$, $x_j \to x$ in X implies $\iota(x_n)f \to \iota(x)f$, i.e. $\sum_j f(t_j(x_n)) \to \sum_j f(t_j(x))$.
- By the transversality of the section, we have $t_{j+1}(x) - t_j(x) \geq \epsilon$ for all $j \in \mathbb{Z}$ and $x \in X$. Hence the above sums have at most K terms, where K only depends on the support of f, not on j, x_n or x.
- It is therefore sufficient to show $f(t_j(x_n)) \to f(t_j(x))$ for each fixed j. This follows from the continuity of f and the continuity of φ^t.

Example: Hitting times for flows

- Fix x_0 and define our deterministic sequence of hitting times by $a_n = t_n(x_0)$, $n = 1, 2, 3. \dots$
- Let t be uniformly distributed in $[0, T]$, and x random with distribution ν; set

$$\xi_T = \sum_{n=1}^{\infty} \delta_{a_n - t}, \qquad \xi = \sum_{j \in \mathbb{Z}} \delta_{t_j(x)}.$$

- The following asserts that the sequence of hitting times converges to a limiting point process (and so in particular yields the convergence of the void statistics):

Theorem 2: Let (φ^t, ν) be ergodic and assume $\nu\left(\bigcup_{-\epsilon \leq t \leq \epsilon} \varphi^t(\partial Y) \right) = 0$. Then, for ν-a.e. $x_0 \in X$,

$$\xi_T \xrightarrow{\text{d}} \xi.$$

as $T \to \infty$.

Example: Hitting times for flows

Proof:

- Define the probability measure ν_{T,x_0} on X by $\nu_T f = \frac{1}{T}\int_0^T f(\varphi^t x_0)dt$ for $f \in \mathsf{C}(X)$
- By the Birkhoff ergodic theorem, for ν-a.e. x_0

$$\nu_{T,x_0} \xrightarrow{\mathsf{w}} \nu$$

which in turn can be written in terms of the random variables $t \in [0, T]$ and $x \in X$ as

$$\varphi^t x_0 \xrightarrow{\mathsf{d}} x.$$

- The measure-zero assumption on the boundary implies $\nu S = 0$ (as S is a countable union of measure-zero sets), hence the continuous mapping theorem implies in view of Theorem 1

$$\iota(\varphi^t x_0) \xrightarrow{\mathsf{d}} \iota(x).$$

- Complete proof by recalling $\iota(\varphi^t x_0) = \xi_T$ and $\iota(x) = \xi$.

Example: Hitting times for flows

Exercise 6: State and prove the analogue of Theorem 2 for the pigeon hole statistics, assuming now that (φ, ν) be ergodic. (Here $\varphi = \varphi^1$ is the time-one map of the flow φ^t.)

Palm distribution for a stationary random measure

- $\xi \in \mathcal{M}(\mathbb{R})$ a stationary random measure, $B \subset \mathbb{R}$ is a given Borel set with $\mathbb{E}\xi B > 0$.

- The **Palm distribution** Q_ξ corresponding to ξ is defined by
$$Q_\xi f = \frac{1}{\mathbb{E}\xi B}\,\mathbb{E}\int_B f(\theta^u \xi)\,\xi(du),$$
with $f : \mathcal{M}(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ measurable. Since ξ is stationary this definition is independent of the choice of B.

- The Palm distribution Q_ξ defines a new random measure $\eta \in \mathcal{M}(\mathbb{R})$ via $\mathbb{E}f(\eta) = Q_\xi f$.

- This definition can be extended to non-stationary random measures ξ.

Palm distribution for stationary point processes

- If ξ is a stationary point process, we can write $\xi = \sum_j \delta_{\tau_j(\xi)}$, and so

$$\mathbb{E}f(\eta) = \frac{1}{I_\xi \mathrm{Leb} B} \mathbb{E} \sum_j \mathbb{1}(\tau_j(\xi) \in B) f\Big(\sum_i \delta_{\tau_i(\xi) - \tau_j(\xi)} \Big).$$

- This shows that η is a point process and furthermore that, if ξ is a simple point process, then η is a simple point process and $\eta\{0\} = 1$ a.s.
- The stationarity of ξ implies that η is **cycle-stationary**; that is $\eta = \sum_i \delta_{\tau_i(\eta)}$ has the same distribution as the point process $\theta^{\tau_j(\eta)}\eta = \sum_i \delta_{\tau_i(\eta) - \tau_j(\eta)}$ for any j.
- The intensity measure of a Palm distributed ζ is in fact the pair correlation measure

$$\mathbb{E}\eta = \frac{1}{I_\xi \mathrm{Leb} B} \mathbb{E} \sum_{i,j} \mathbb{1}(\tau_j(\xi) \in B) \delta_{\tau_i(\xi) - \tau_j(\xi)}$$

(up to the additional δ_0 from $i = j$)

Example: Poisson point processes

Exercise 7: Show that if ξ is a homogeneous Poisson process with intensity I_ξ, then $\eta \overset{\mathrm{d}}{=} \delta_0 + \xi$.

- This relation is in fact unique to the Poisson process (Slivnyak's theorem): If ξ is a stationary process on $\mathbb{R}$ and $\delta_0 + \xi$ is distributed according to Q_ξ, then ξ is a homogeneous Poisson process.

Example: Return times for flows

- As above, consider the topological flow $\varphi^t : X \to X$ where (X, ν) is a probability space and ν invariant under φ^t.
- Choose a section $Y \subset X$ that is transversal to the flow, and denote by μ the invariant measure on for the return map
- For $x \in X$, let $(t_j(x))_{j\in\mathbb{Z}}$ be the sequence of **hitting times**
- In the special case $x \in Y$, we call $(t_j(x))_{j\in\mathbb{Z}}$ be the sequence of **return times**
- For $x \in X$ random with distribution ν and $y \in Y$ random with distribution μ, set

$$\xi = \sum_j \delta_{t_j(x)}, \qquad \eta = \sum_j \delta_{t_j(y)}.$$

- One can show that η **is distributed according to the Palm distribution** Q_ξ **of** ξ**.***

*J. Marklof, Nonlinearity 2019; goes back to Ambrose and Kakutani's work in the 1940's

Palm inversion theorem

Theorem 3: Assume ξ is a simple, stationary point process on $\mathbb{R}$ with positive finite intensity. Let η be a point process distributed according to Q_ξ. Then $\mathbb{P}(\xi \in \cdot \mid \xi \neq 0)$ is uniquely determined by η, and, for any measurable $f : \mathcal{N}(\mathbb{R}) \to \mathbb{R}_{\geq 0}$,

$$\mathbb{E}[f(\xi)\mathbb{1}(\xi \neq 0)] = I_\xi \, \mathbb{E} \int_0^{\tau_1(\eta)} f(\theta^u \eta) du.$$

- Note that the theorem yields for $f \equiv 1$ the relation $\mathbb{E}\mathbb{1}(\xi \neq 0) = I_\xi \mathbb{E}\tau_1(\eta)$.
- Furthermore the choice $f(\zeta) = \mathbb{1}(\tau_1(\zeta) > R)$ yields...

Palm-Khinchin equations

- Furthermore the choice $f(\zeta) = \mathbb{1}(\tau_1(\zeta) > R)$ yields

$$\begin{aligned}\mathbb{P}(\tau_1(\xi) > R \mid \xi \neq 0) &= \frac{1}{\mathbb{E}\tau_1(\eta)}\mathbb{E}\int_0^{\tau_1(\eta)} \mathbb{1}(\tau_1(\eta) - u > R)du \\ &= \frac{1}{\mathbb{E}\tau_1(\eta)}\mathbb{E}\int_0^{\infty} \mathbb{1}(\tau_1(\eta) > R + u)du\end{aligned}$$

  and so

$$\mathbb{P}(\tau_1(\xi) > R \mid \xi \neq 0) = \frac{1}{\mathbb{E}\tau_1(\eta)}\int_R^{\infty} \mathbb{P}(\tau_1(\eta) > u)du.$$

- *Does this look familiar?*

Exercise 8: Prove analogous relations for $\tau_j(\xi)$, $j > 1$.
(These are known as **Palm-Khinchin equations**.)

Convergence

Theorem 4: Let (ξ_T) be a sequence of stationary point processes on $\mathbb{R}$ with $0 < I_{\xi_T} < \infty$, and η_T a point process given by the Palm distribution of ξ_T. Then any two of the following statements imply the third:

(i) $I_{\xi_T} \to I_\xi$;

(ii) $\xi_T \xrightarrow{\mathrm{d}} \xi$;

(iii) $\eta_T \xrightarrow{\mathrm{d}} \eta$, where η has distribution Q_ξ.

- In fact also holds in more general form for non-stationary processes*
- This in particular implies that the convergence of the void statistics implies the convergence of the gap statistics and vice versa (can also be proved directly), and in the context of dynamical systems that the convergence of the hitting time process implies the convergence of the return time process and vice versa†

*O. Kallenberg, Zeitsch. Wahrsch. Theo. Verw. Geb. 1973

†N. Haydn, Y. Lacroix, S. Vaienti, Ann. Probab. 2005; R. Zweimüller, Israel Math J. 2016; J. Marklof, Nonlinearity 2017

The stationarity trick

- Issue: $\xi_T = \sum_{n=1}^{\infty} \delta_{a_n - t}$ is not a stationary point process; we assume here t is uniformly distributed in $[0, T]$

- Consider instead $\tilde{\xi}_T = \sum_{m \in \mathbb{Z}} \sum_{n=1}^{\infty} \mathbb{1}(0 \leq a_n < T)\, \delta_{a_n + Tm - t}$

> **Exercise 9**:
> (i) Show $\tilde{\xi}_T$ is a stationary point process.
> (ii) Show that $\tilde{\xi}_T \xrightarrow{\mathrm{d}} \xi$ if and only if $\xi_T \xrightarrow{\mathrm{d}} \xi$.

- $\tilde{\xi}_T$ in fact arises naturally in the fine-scale statistics of sequences modulo one; more on that later

Case study 1: Hitting and return times for linear flows on flat tori

Based on J. Marklof, A. Strömbergsson, Annals Math. 2010

Linear flows on flat tori

- $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ the standard d-dimensional torus

- $(q, v) \in \mathbb{T}^d \times \mathsf{S}_1^{d-1}$ = phase space of position and velocity

- Linear flow $\varphi^t(q, v) = (q + tv, v)$; preserves Lebesgue measure

- If the coefficients of v are linearly independent over $\mathbb{Q}$ then for any $f \in \mathsf{C}(\mathbb{T}^d)$
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\varphi^t(q, v))dt = \int_{\mathbb{T}^d} f(x, v)dx$$
(Kronnecker-Weyl theorem)

- Similar statement for all v but equidistribution on rational embedded subtori

Hitting and return times

- $D \subset \mathbb{R}^{d-1}$ bounded Borel set with boundary of measure zero. Embed in $\mathbb{R}^d$ as $\{0\} \times D$; will abbreviate this as $D \subset \mathbb{R}^d$
- Assume D has diameter < 1; this ensures that $(D + \mathbb{Z}^d R) \cap D = \emptyset$ for all $R \in \mathrm{SO}(d)$.
- Fix any piecewise smooth map $K : \mathrm{S}_1^{d-1} \to \mathrm{SO}(d)$ so that $vK(v) = e_1 = (1, 0, \ldots, 0)$

For example, we may choose K as $K(e_1) = I$, $K(-e_1) = -I$ and $K(v) = E\Big(-\frac{2\arcsin\big(||v - e_1||/2\big)}{||v_\perp||} v_\perp\Big)$ for $v \in \mathrm{S}_1^{d-1} \setminus \{e_1, -e_1\}$, where $v_\perp := (v_2, \ldots, v_d) \in \mathbb{R}^{d-1}$ and $E(w) = \exp\begin{pmatrix} 0 & w \\ -{}^{t}w & 0 \end{pmatrix} \in \mathrm{SO}(d)$. Then K is smooth when restricted to $\mathrm{S}_1^{d-1} \setminus \{-e_1\}$.

Hitting and return times

- Define the section $Y = \{(q,v) \mid q \in DK(v)^{-1},\ v \in \mathsf{S}_1^{d-1}\} \subset X$; note Y is a transversal section for the flow φ^t
- The sequence of hitting times $(t_j(q,v))_j$ is given by the set

$$\{t \in \mathbb{R} \mid q + tv \in DK(v)^{-1} + \mathbb{Z}^d\}$$

- Define the cylinder $\mathcal{Z}(D) = \mathbb{R} \times D = \{(t,y) \mid t \in \mathbb{R},\ y \in D\}$.
- Let π_1 denote the orthogonal projection $\pi_1 : \mathbb{R}^d \to \mathbb{R}$, $x \mapsto e_1 \cdot x$.

Exercise 10: Show that the sequence of hitting times $(t_j(q,v))_j$ is given by the set*

$$\pi_1\Big(\mathcal{Z}(-D) \cap \big[(\mathbb{Z}^d - q)K(v)\big]\Big).$$

*This is in fact a **cut-and-project set/Euclidean model set** known from the construction of "quasicrystals"

Hitting times for shrinking sections

- Now fix $D \subset \mathbb{R}^{d-1}$ an open bounded Borel set with boundary of measure zero, and consider the shrinking sections $D_r = rD$ with $r \to 0$.
- Does the sequence of hitting times $(t_j^{(r)}(q,v))_j$ for the section $D_r = rD$ converge to a limit process, for (q,v) suitably random?

Exercise 11: Let $q \in D$. Show that $(t_j^{(r)}(q,v))_j$ is given by the set

$$r^{1-d}\, \pi_1\Big(\mathcal{Z}(-D) \cap \big[(\mathbb{Z}^d - q) K(v) A(r) \big] \Big)$$

with $A(r) = \mathrm{diag}(r^{d-1}, r^{-1}, \ldots, r^{-1}) \in \mathrm{SL}(d, \mathbb{R})$.

Return times for shrinking sections

- Note that for $q = rbK(v)^{-1}$ with $b \in \{0\} \times D$ parametrising the section, we have $(\mathbb{Z}^d - q)K(v)A(r) = \mathbb{Z}^d K(v)A(r) - b$

- So in this case $(t_j^{(r)}(q,v))_j$ is given by

$$r^{1-d}\,\pi_1\Big(\mathcal{Z}(-D+b) \cap \big[\mathbb{Z}^d K(v)A(r)\big]\Big)$$

The space of lattices

- $G_0 = \mathsf{SL}(d, \mathbb{R})$, $\Gamma_0 = \mathsf{SL}(d, \mathbb{Z})$.

- The map $\Gamma_0 M \mapsto \mathbb{Z}^d M$ gives a one-to-one correspondence between the homogeneous space $\Gamma_0 \backslash G_0$ and the space of Euclidean lattices in $\mathbb{R}^d$ of covolume one.

- The Haar measure μ_0 on G_0 is normalized so that it gives a probability measure on $\Gamma_0 \backslash G_0$; also denote by μ_0

The space of affine lattices

- $G = G_0 \ltimes \mathbb{R}^d$ the semidirect product with multiplication law
$$(M, z)(M', z') = (MM', zM' + z')$$

- Define action of $g = (M, z) \in G$ on $\mathbb{R}^d$ by $yg = yM + z$.

- $\Gamma = \Gamma_0 \ltimes \mathbb{Z}^d$ is a lattice in G.

- The Haar measure on G is $\mu = \mu_0 \times \mathrm{Leb}$ (the Lebesgue measure normalised so that $\mathrm{Leb}[0,1]^d = 1$); corresponding probability measure on $\Gamma \backslash G$ also denoted by μ.

- We embed G_0 in G via $M \mapsto (M, 0)$.

- We embed X_0 in X via $\Gamma_0 M \mapsto \Gamma(M, 0)$.

Equidistribution

Theorem 5*: For $f : X \to \mathbb{R}$ bounded continuous, λ absolutely continuous Borel probability measure on S_1^{d-1}, and $r \to 0$,

$$\int_{\mathsf{S}_1^{d-1}} f(\Gamma(1,q)K(v)A(r))\lambda(dv) \to \begin{cases} \nu f & \text{if } q \notin \mathbb{Q}^d \\ \nu_0 f & \text{if } q = 0 \end{cases}$$

- Let us think of $v \in \mathsf{S}_1^{d-1}$ as a random variable with distribution λ, and define the random element $x_{r,q} = \Gamma(1,q)K(v)A(r) \in X$.
- Then the theorem can be restated as

$$x_{r,q} \xrightarrow{\mathrm{d}} \begin{cases} x & \text{if } q \notin \mathbb{Q}^d \\ x_0 & \text{if } q = 0 \end{cases}$$

where x and x_0 are random elements with distribution ν and ν_0, respectively.

*Follows from Ratner's measure classification theorem

Random lattices as point processes

Theorem 6: The map

$$\iota\colon X \to \mathcal{M}(\mathbb{R}^d), \qquad x \mapsto \sum_{y \in \mathbb{Z}^d x} \delta_y .$$

is a topological embedding.*

- The key point we need from this statement is the **continuity** of ι, which is proved as follows: We need to show that, for every $f \in \mathrm{C}_c(\mathbb{R}^d)$, $x_j \to x$ in X implies $\iota(x_j)f \to \iota(x)f$. By the Γ-equivariance of ι, it is sufficient to show that $g_j \to g$ in G implies $\displaystyle\sum_{y \in \mathbb{Z}^d g_j} f(y) \to \sum_{y \in \mathbb{Z}^d g} f(y)$. Let A be the compact support of f. Since $g_j \to g$, the closure of $A' = \cup_j (A g_j^{-1})$ is compact. Hence $\mathbb{Z}^d \cap A'$ is finite. For $a \in \mathbb{Z}^d \setminus A'$ we have $f(ag_j) = f(ag) = 0$, and for the finitely many $a \in \mathbb{Z}^d \cap A'$ we have $f(ag_j) \to f(ag)$. QED

*That is, ι is a continuous injection which gives a homeomorphism $X \to \iota(X)$, where $\iota(X) \subset \mathcal{M}(\mathbb{R}^d)$ is equipped with the subspace topology. See J. Marklof, I. Vinogradov, Geom. Dedicata 2017 for a full proof of the theorem.

Random lattices as point processes

Theorem 6: The map

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- The continuous mapping theorem will allow us now to convergence statements on X, X_0 to limit theorems for the corresponding point processes:

$$\iota(x_{r,q}) \xrightarrow{\mathrm{d}} \begin{cases} \iota(x) & \text{if } q \notin \mathbb{Q}^d \\ \iota(x_0) & \text{if } q = 0 \end{cases}$$

- This yields in particular the desired limit theorem for the hitting an return times...

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Siegel's formula

- For any Borel measurable $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$

$$\int_{X_0} \left(\sum_{y \in \mathbb{Z}^d \tilde{x}} f(y) \right) \nu_0(d\tilde{x}) = f(0) + \int_{\mathbb{R}^d} f(y)\, dy$$

- We can restate this as a formula for the intensity measure of the point process $\iota(x_0)$:

$$\mathbb{E}\iota(x_0) = \delta_0 + \mathrm{Leb}$$

> **Exercise 12**: Show that the point process $\iota(x)$ is stationary and its intensity measure is $\mathbb{E}\iota(x) = \mathrm{Leb}$.

- The point process $\iota(x_0)$ is in fact distributed according to the Palm distribution of $\iota(x)$

Limit theorem for hitting/return times for shrinking target

Theorem 7*: (i) For $q \notin \mathbb{Q}$

$$\sum_j \delta_{r^{d-1} t_j^{(r)}(q,v)} \xrightarrow{\mathrm{d}} \xi = \sum_{y \in \mathcal{Z}(-D) \cap \mathbb{Z}^d x} \delta_{\pi_1(y)}$$

with random $x \in X$ with distribution ν.

(ii) For $q = rbK(v)^{-1}$

$$\sum_j \delta_{r^{d-1} t_j^{(r)}(q,v)} \xrightarrow{\mathrm{d}} \eta_b = \sum_{y \in \mathcal{Z}(-D+b) \cap \mathbb{Z}^d x_0} \delta_{\pi_1(y)}.$$

with random $x_0 \in X_0$ with distribution ν_0.

*J. Marklof, A. Strömbergsson, Annals Math 2010 [in dimension $d = 2$ Boca, Zaharescu (Comm. Math. Phys. 2007) proved convergence of first hitting time $r^{d-1} t_j^{(r)}(q, v)$ including explicit formula for limit distribution; see also P. Dahlqvist, Nonlinearity 1997]

Proof:

- We can write $\Xi \in \mathcal{N}(\mathbb{R}^d)$ as $\Xi = \sum_j \delta_{\mathcal{T}_j(\Xi)}$ with $\mathcal{T}_j : \mathcal{N}(\mathbb{R}^d) \to \mathbb{R}^d$.
- Note that the map

$$\kappa_D : \mathcal{N}(\mathbb{R}^d) \to \mathcal{N}(\mathbb{R})$$

$$\sum_j \delta_{\mathcal{T}_j(\xi)} \mapsto \sum_j \mathbb{1}(\mathcal{T}_j(\Xi) \in \mathcal{Z}(-D))\, \delta_{\pi_1(\mathcal{T}_j(\Xi))}$$

  is continuous outside the closed subset

$$S = \{\Xi \in \mathcal{N}(\mathbb{R}^d) \mid \Xi(\partial \mathcal{Z}(-D)) \geq 1\}.$$

- We have

$$\iota(x)S \leq \mathbb{E}\mathbb{1}\Big(\Xi(\partial \mathcal{Z}(-D)) \geq 1\Big) \leq \mathbb{E}\Big(\Xi(\partial \mathcal{Z}(-D))\Big) = \mathrm{Leb}(\partial \mathcal{Z}(-D)) = 0.$$

  and similalrly

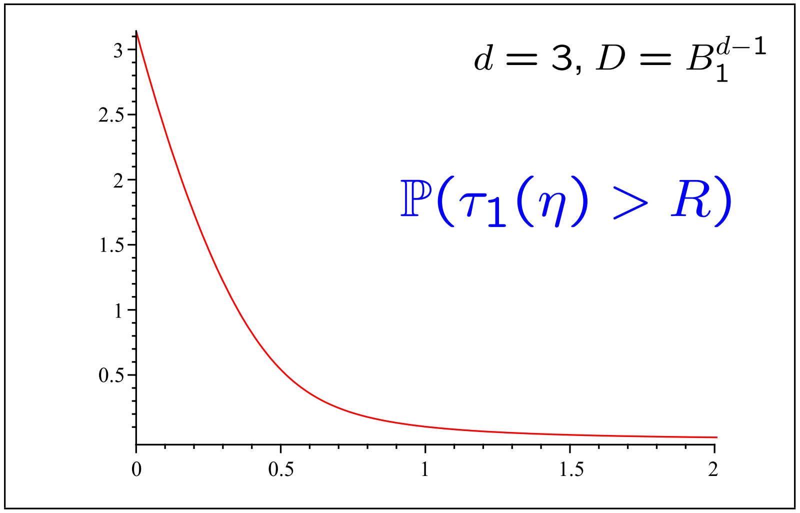
$$\iota(x_0)S \leq \delta_0(\partial \mathcal{Z}(-D+b)) + \mathrm{Leb}(\partial \mathcal{Z}(-D+b)) = 0.$$

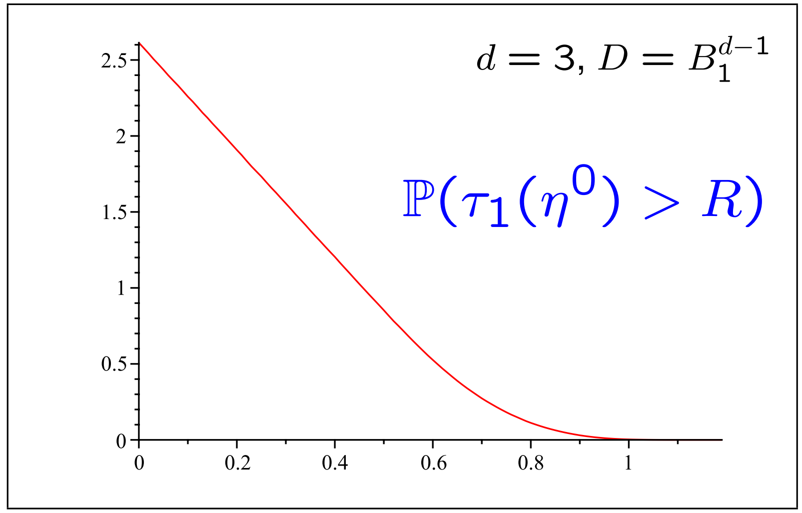
  We have used here that $b \in D$ and hence $0 \in \mathcal{Z}(-D+b)$. So $0 \notin \partial \mathcal{Z}(-D+b)$ since D is assumed open.
- Now apply continuous mapping theorem.

Limit theorem for hitting/return times for shrinking target

- It follows from the stationarity of $\iota(x)$ that ξ is stationary.

> **Theorem 8***: Assume b is uniformly distributed in D and let $\eta = \eta^b$ be the corresponding point process. Then η is distributed according to the Palm distribution Q_ξ.





- Tail asymptotics† $-\dfrac{d\mathbb{P}(\tau_1(\eta) > R)}{dR} \sim \dfrac{A_d}{R^3}$ with $A_d = \dfrac{2^{2-d}}{d(d+1)\zeta(d)}$

*J. Marklof, A. Strömbergsson, Annals Math 2010

†J. Marklof, A. Strömbergsson, GAFA 2011

Case study 2: Fractional parts of $\sqrt{n}$

Based on: N. Elkies, C. McMullen, Duke. Math. J. 2004*

*See also: J. Marklof, Distribution modulo one and Ratner's theorem, Equidistribution in Number Theory, An Introduction, Springer 2007

Triangular arrays

- Consider the triangular array

$$\begin{array}{lllll} \alpha_{11} & & & & \\ \alpha_{21} & \alpha_{22} & & & \\ \vdots & \vdots & \ddots & & \\ \alpha_{N1} & \alpha_{N2} & \cdots & \alpha_{NN} & \\ \vdots & \vdots & & & \ddots \end{array}$$

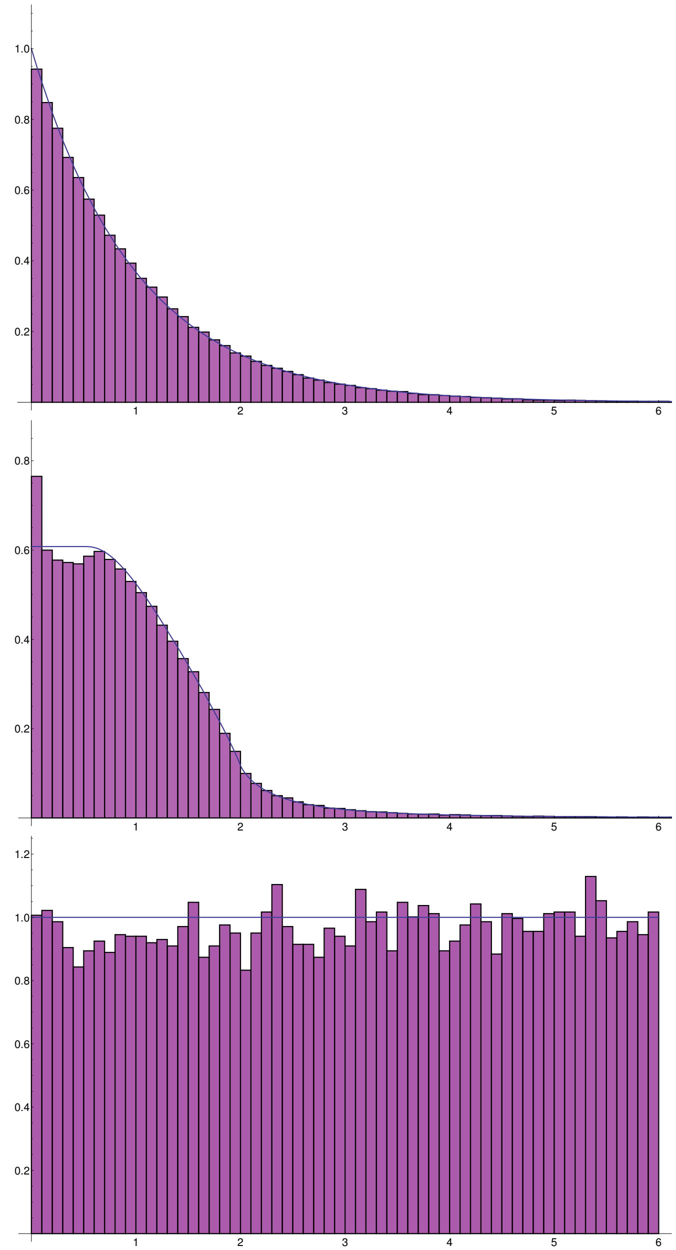
  with $\alpha_{Nn} \in [0,1)$ such that $\alpha_{Nn} \leq \alpha_{N,n+1}$
- We say (α_{Nn}) is **uniformly distributed mod 1** if for $0 \leq a < b \leq 1$

$$\lim_{N\to\infty} \frac{\#\{n \leq N : \alpha_{Nn} \in [a,b] + \mathbb{Z}\}}{N} = b - a.$$

- Want to study **fine-scale statistics of such triangular arrays mod 1**
- **Example:** Take α_{Nn} to be the fractional parts of $(n^\beta)_{n=1}^N$, with $0 < \beta < 1$ fixed.

Fractional parts of small powers

- For fixed $0 < \beta < 1$, $\beta \neq \dfrac{1}{2}$, the gap and two-point statistics of $n^{\beta} \bmod 1$ look Poisson numerically—-NO PROOFS! $\beta = \dfrac{1}{3} \rightarrow$
- For $\beta = \dfrac{1}{2}$, Elkies & McMullen (Duke Math J 2004) have shown that the gap distribution exists, and derived an explicit formula which is clearly different from the exponential. Their proof uses Ratner's measure classification theorem!
- At the same time, the two-point function converges to the Poisson answer (with El Baz & Vinogradov, Proc AMS 2015). The proof requires upper bounds for the equidistribution of certain unipotent flows with respect to unbounded test functions.



Sequence of sequences

- To connect with our previous setting, define for each N the sequence

$$-\infty \leftarrow \ldots \leq a_{N,-1} \leq a_{N,0} < 0 \leq a_{N1} \leq a_{N2} \leq \cdots \to \infty$$

given by

$$a_{N,n+Nm} = N\alpha_{Nn} + Nm, \qquad n = 1, \ldots, N, \; m \in \mathbb{Z}.$$

- Previously we dealt with a fixed sequence (a_n) of non-negative elements, now it is (a_{Nn}), a **sequence*** **of bi-infinite sequences**†—no problem! (Recall the stationarity trick)

*indexed by $N \in \mathbb{N}$

†indexed by $n \in \mathbb{Z}$

Point processes

- Fix as before σ a locally finite Borel measure on $\mathbb{R}_{\geq 0}$ so that $\sigma[0, \infty) = \infty$.

- We are interested in the sequence of point processes (cf. "randomisation" slide)

$$\xi_N = \sum_{n \in \mathbb{Z}} \delta_{a_{Nn} - t}$$

- Here t is a random variable distributed on $[0, N)$ with respect to σ; that is t is defined by $\mathbb{P}(t \in B) = \dfrac{\sigma(B \cap [0, N))}{[0, N)}$ for any Borel set $B \subset \mathbb{R}$.

- Note that if $\sigma = \text{Leb}$, then ξ_N is stationary.

- If $\sigma = \sum_{n \in \mathbb{Z}} \delta_{a_{Nn}}$ then ξ_N is cycle stationary (and distributed according to the Palm distribution of the previous example).

Fractional parts of $\sqrt{n}$

- Take α_{Nn} to be the fractional parts of $(\sqrt{n})_{n=1}^{N}$
- The sequence $(a_{Nn} - t)_n$ is then given by the ordered set (put $t = Ns$)

$$P_{N,s} = \{N(\sqrt{n} + m - s)) \mid n = 1, \ldots, N,\ m \in \mathbb{Z}\}$$

- “Lift” this to the following point set in $\mathbb{R}^2$:

$$Q_{N,s} = \left\{ \left(\frac{n^{1/2}}{N^{1/2}}, N\left(n^{1/2} + m - s\right) \right) \,\middle|\, (m,n) \in \mathbb{Z}^2,\ n > 0 \right\}$$

  and note that $P_{N,s} = \pi_2\left[Q_{N,s} \cap \left((0,1] \times \mathbb{R}\right)\right]$ (cut and project!).
- Here is another point set in $\mathbb{R}^2$:

$$\widetilde{Q}_{N,s} = \left\{ \left(\frac{m+s}{N^{1/2}}, -\frac{N^{1/2}(n + 2ms + s^2)}{2N^{-1/2}(m+s)} \right) \,\middle|\, (m,n) \in \mathbb{Z}^2 \right\}$$

- $Q_{N,s}$ and $\widetilde{Q}_{N,s}$ are close (in the right half plane) . . .

The key observation

- Fix any compact set $\mathcal{A} \subset \mathbb{R}_{>0} \times \mathbb{R}$. Then for any element in $Q_{N,s} \cap \mathcal{A}$ we have $n^{1/2} = -m + s + O_{\mathcal{A}}(N^{-1})$, so

$$\begin{aligned}&\left(\frac{n^{1/2}}{N^{1/2}}, N\left(n^{1/2} + m - s\right)\right)\\ &= \left(\frac{n^{1/2}}{N^{1/2}}, \frac{N\left(n - (-m+s)^2\right)}{n^{1/2} - m + s}\right)\\ &= \left(\frac{-m+s}{N^{1/2}} + O_{\mathcal{A}}(N^{-3/2}), \frac{N^{1/2}\left(n - (-m+s)^2\right)}{2N^{-1/2}(-m+s) + O_{\mathcal{A}}(N^{-3/2})}\right)\end{aligned}$$

- Now shift n by m^2 (this 1:1 on $\mathbb{Z}$) and then replace (m,n) by $-(m,n)$. This shows that each element in $Q_{N,t} \cap \mathcal{A}$ is $O(N^{-3/2})$-close to a unique point in

$$\widetilde{Q}_{N,t} = \left\{\left(\frac{m+s}{N^{1/2}}, -\frac{N^{1/2}(n+2ms+s^2)}{2N^{-1/2}(m+s)}\right) \,\middle|\, (m,n) \in \mathbb{Z}^2\right\}.$$

The key observation

Exercise 13: Show that

$$\widetilde{Q}_{N,s} = \left\{ \left(y_1, -\frac{y_2}{2y_1} \right) \;\middle|\; (y_1, y_2) \in \mathbb{Z}^2 P(s) A(N^{-1/2}) \right\}$$

where

$$P(s) = \left(\begin{pmatrix} 1 & 2s \\ 0 & 1 \end{pmatrix}, (s, s^2) \right), \qquad A(r) = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}.$$

- Check that $P(s)$ generates a one-parameter subgroup of $\mathrm{ASL}(2, \mathbb{R})$.

Equidistribution

Theorem 9*: For $f : X \to \mathbb{R}$ bounded continuous, λ absolutely continuous Borel probability measure on $[0,1]$, and $r \to 0$,

$$\int_0^1 f(\Gamma P(s)A(r))\lambda(ds) \to \nu f.$$

- By the same strategy as in the previous section this implies...

*Follows from Ratner's measure classification theorem; for an effective proof see T. Browning, I. Vonogradov, J. LMS 2016, building on the crucial work by A. Strömbergsson, Duke Math. J. 2015

Limit theorem for the $\sqrt{n}$ process

Theorem 10: Let t be a unformly distributed random variable in $[0, T)$. Then

$$\xi_N = \delta_{a_{Nn}-t} \xrightarrow{\text{d}} \xi = \sum_{\substack{(y_1,y_2)\in\mathbb{Z}^d x \\ y_1\in(0,1]}} \delta_{-y_2/2y_1}$$

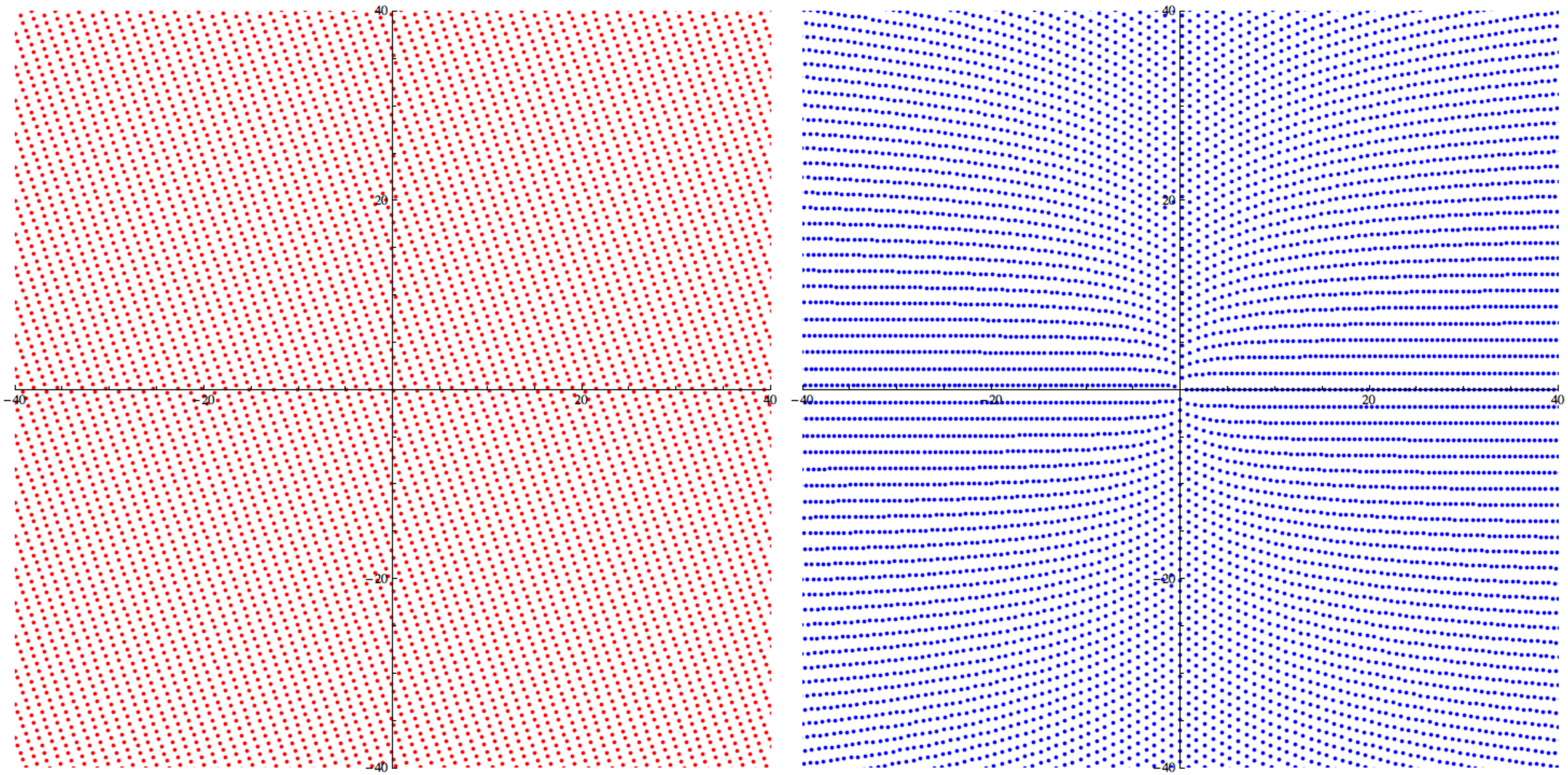
with random $x \in X$ with distribution ν, and for the corresponding Palm distributed processes

$$\eta_N = \xrightarrow{\text{d}} \eta = \sum_{\substack{(y_1,y_2)\in\mathbb{Z}^d x_0+(b,0) \\ y_1\in(0,1]}} \delta_{-y_2/2y_1}$$

with random $x_0 \in X_0$ with distribution ν_0, and b uniformly distributed in $(0, 1]$.

On the home straight, a slightly different perspective on $\sqrt{n}$ mod 1...

Square-roots and lattice points



Lattice points in a Euclidean lattice vs. $\mathcal{P} = \left\{ \left(\sqrt{\frac{n}{\pi}} \cos\left(2\pi\sqrt{n}\right), \sqrt{\frac{n}{\pi}} \sin\left(2\pi\sqrt{n}\right) \right) \,\middle|\, n \in \mathbb{N} \right\}$

Square-roots and lattice points

The statistics of $\sqrt{n} \bmod 1$ is equivalent to the directional statistics of the point set

$$\mathcal{P} = \left\{ \left(\sqrt{\frac{n}{\pi}} \cos\left(2\pi\sqrt{n}\right), \sqrt{\frac{n}{\pi}} \sin\left(2\pi\sqrt{n}\right) \right) \,\middle|\, n \in \mathbb{N} \right\}$$

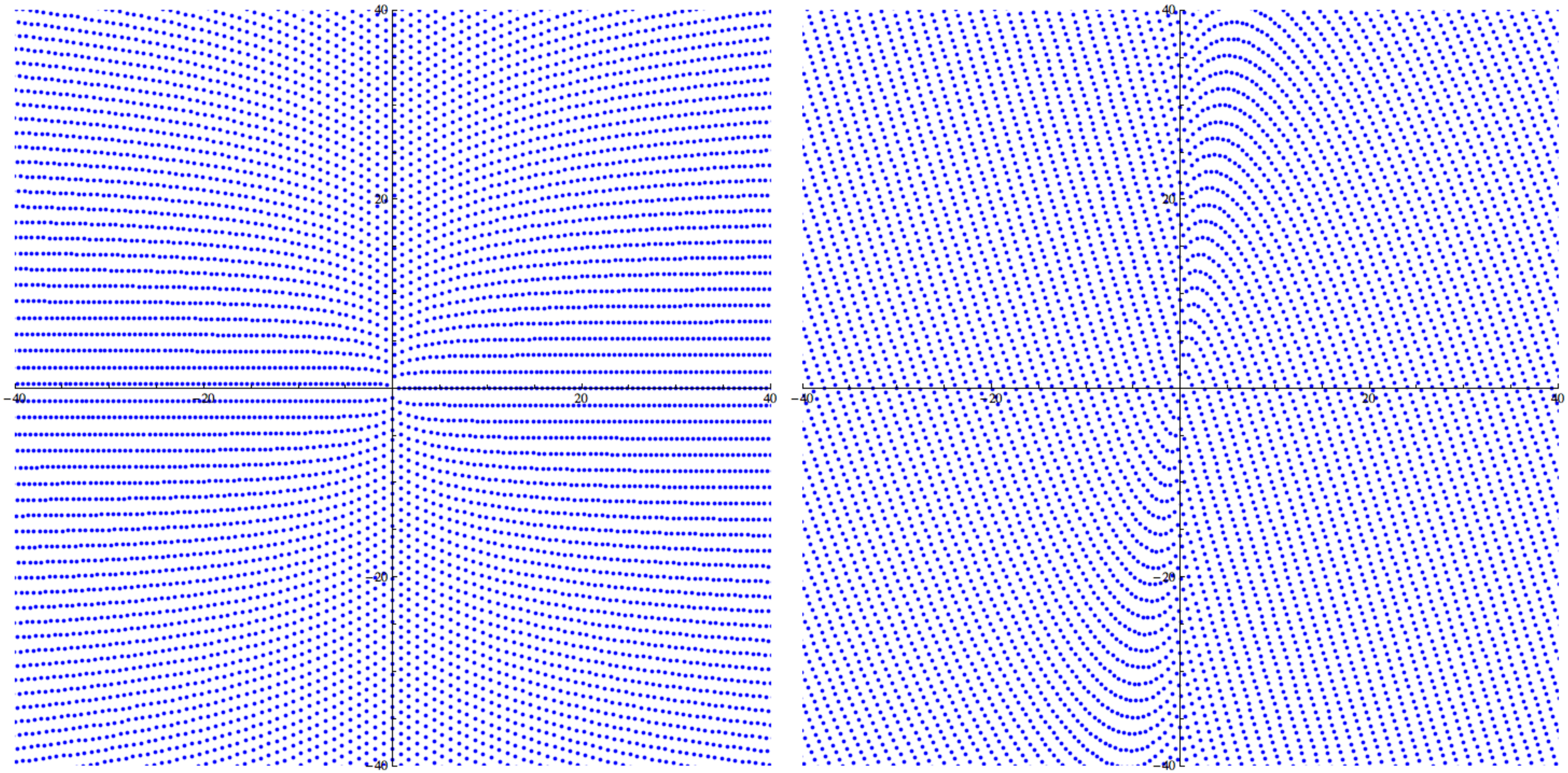
To understand the directional statistics of a point set, we need to rotate and dilate

$$k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad D(T) = \begin{pmatrix} T^{-1/2} & 0 \\ 0 & T^{1/2} \end{pmatrix}$$

which yields

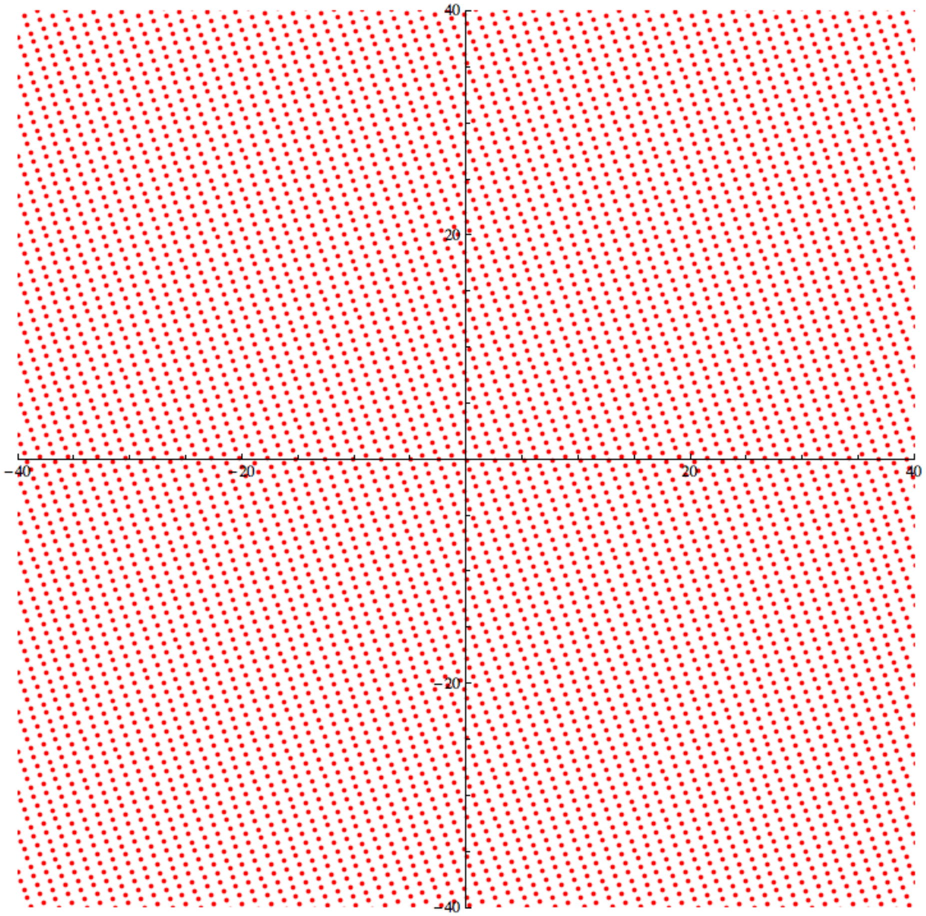
$$\mathcal{P}k(\theta)D(T) = \left\{ \left(\sqrt{\frac{n}{\pi T}} \cos(2\pi\sqrt{n} - \theta), \sqrt{\frac{Tn}{\pi}} \sin(2\pi\sqrt{n} - \theta) \right) \,\middle|\, n \in \mathbb{N} \right\}$$

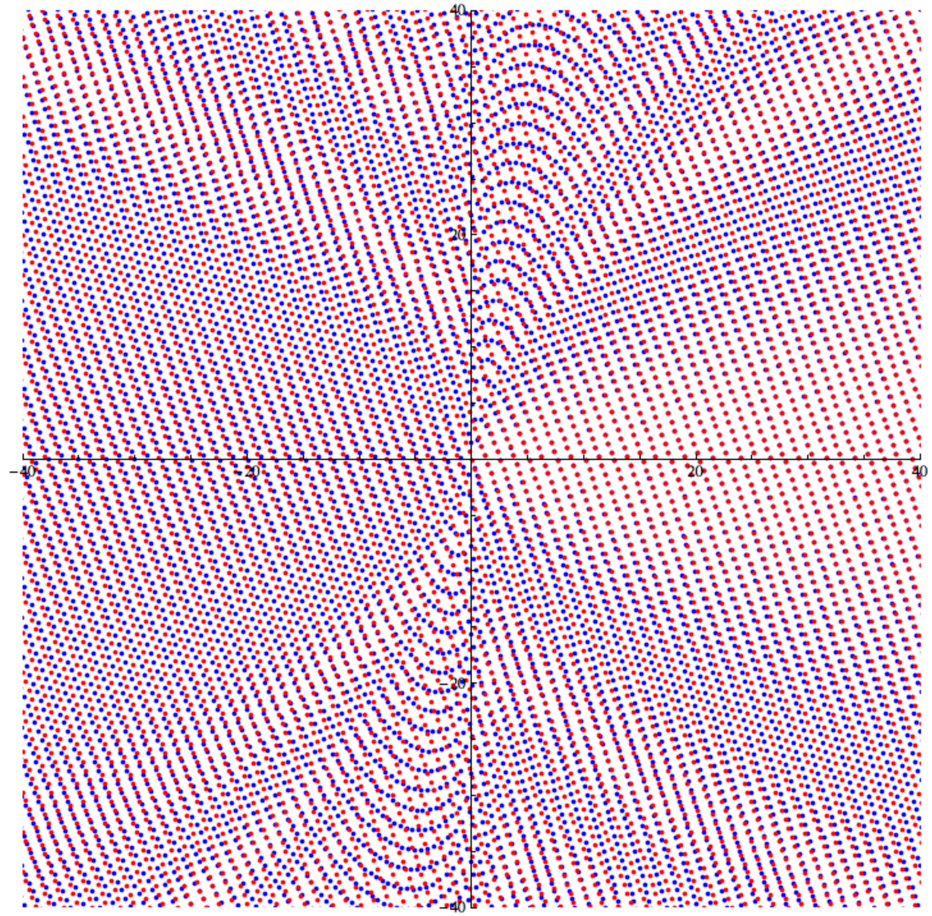
Square-roots and lattice points



The point sets $\mathcal{P}$ and $\mathcal{P}k(\theta)D(T)$ with $T = 4$ and $\theta = 0.7$.

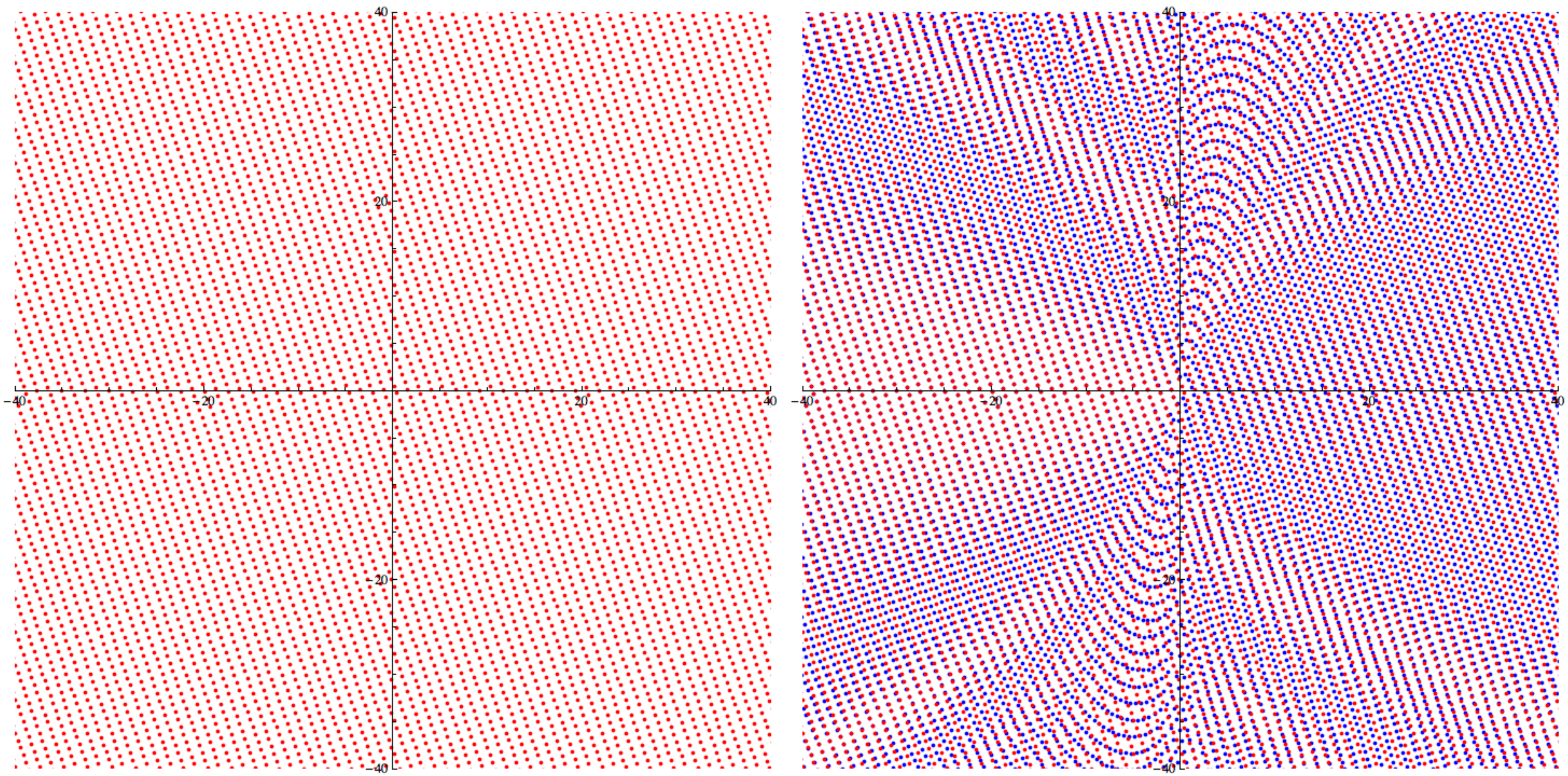
Square-roots and lattice points





The approximation of $\mathcal{P}k(\theta)D(T)$ by an affine lattice in fixed bounded subsets of the right halfplane.

Square-roots and lattice points



The approximation of $\mathcal{P}k(\theta)D(T)$ by an affine lattice in fixed bounded subsets of the left halfplane.

Further reading

- Gap distributions for sequences mod 1: J. Marklof, Distribution modulo one and Ratner's theorem, Equidistribution in Number Theory, An Introduction, eds. A. Granville and Z. Rudnick, Springer 2007, pp. 217-244.
- Linear flows and much more: J. Marklof and A. Strömbergsson, The distribution of free path lengths in the periodic Lorentz gas and related lattice point problems, Annals of Mathematics 172 (2010) 1949-2033
- For Palm distribution and dynamics: J. Marklof, Entry and return times for semi-flows, Nonlinearity 30 (2017) 810-824.
- Return maps for the horocyle flow: J. Athreya and Y. Cheung, A Poincaré section for the horocycle flow on the space of lattices. Int. Math. Res. Not. IMRN 2014, 2643-2690.
- (What we did not have time for) Hyperbolic lattice points: J. Marklof and I. Vinogradov, Directions in hyperbolic lattices, Journal für die Reine und Angewandte Mathematik 740 (2018) 161-186